$T^\alpha = 1$; *i. e.* a transform of T cannot be of a lower order than T. Since the order of any transform of T is equal to the order of T it follows that α must be prime to the order of T in $S^{-1}TS = T^\alpha$. By writing any such power of T below T we can at once write down the substitution which transforms T into this power. Hence α can have all the values that are prime to the order of T. To complete the determination of the order of G it is desirable to arrange its substitutions in a rectangular form, as follows:*

$$\begin{array}{llll} 1 & s_1 & s_2 \ldots\ldots\ldots & s_{l-1} \\ t_1 & s_1 t_1 & s_2 t_1 \ldots\ldots & s_{l-1} t_1 \\ t_2 & s_1 t_2 & s_2 t_2 \ldots\ldots & s_{l-1} t_2 \\ \cdot & \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot & \cdot \\ \cdot & \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot & \cdot \\ t_{k-1} & s_1 t_{k-1} & s_2 t_{k-1} \ldots\ldots & s_{l-1} t_{k-1} \end{array}$$

The first row contains all the substitutions which transform T into itself and hence forms a subgroup of G. It is supposed that each t is so selected that it does not occur in any row that precedes it. Since the first row includes all the substitutions of G that transform T into itself $t_1^{-1} T t_1 = T^{\alpha_1}$ (α_1 is not equal to 1). Hence $(s_\alpha t_1)^{-1} T s_\alpha t_1 = t_1^{-1} s_\alpha^{-1} T s_\alpha t_1 {}^* = t_1 T t_1 = T^{\alpha_1}$ ($\alpha_1 = 0, 1, 2, \ldots, l-1$); *i. e.* each of the substitutions of the second row transforms T into T^{α_1}. G cannot contain any other substitution h which has this property for by transforming by t_1^{-1} each member of the equation $h^{-1} T h = T^{\alpha_1}$ we have $(h t_1^{-1})^{-1} T h t_1^{-1} = t_1 T^{\alpha_1} t_1^{-1} = T$. From the fact that $h t_1^{-1} = s_\alpha$ it follows that $h = s_\alpha t_1$. No two of the substitutions of the second row are identical for from $s_\alpha t_1 = s_\beta t_1$ there results $s_\alpha t_1 t_1^{-1} = s_\beta t_1 t_1^{-1}$ or $s_\alpha = s_\beta$. The number of substitutions that transform T into T^{α_1} is therefore equal to l, the number that are commutative with T. The order of G is then kl, k being equal to the number of positive integers less than the order of T and prime to it.

Having determined the order of the groups which are composed of all the substitutions that are commutative with a given substitution T or with the group generated by T, we proceed to determine some of the properties of these groups. We first observe that when T is regular the group which is composed of all the substitutions which transform T into itself contains no selfconjugate substitutions except the powers of T. Since such a group contains the product of all the cycles of T a selfconjugate subgroup could not permute any of the cycles of T, and among the substitutions that do not permute any of these cycles the powers of T are clearly the only ones that are selfconjugate. This fact leads to the following interesting result: *The necessary and sufficient condition that the group of degree n which is composed of all the substitutions which are commutative to T is*

*This arrangement seems to have been first employed by Abbati in 1802, *Burkhardt, Zeitschrift fur Mathematik und Physik*, Vol. 37, page 141. By means of it we can see directly that the order of any subgroup is a divisor of the order of the group.

†$(t_1 t_2 \ldots t_r)^{-1} = t_r^{-1} \ldots t_2^{-1} t_1^{-1}$ since $t_1 t_2 \ldots t_r t_r^{-1} \ldots t_2^{-1} t_1^{-1} = 1$.

conjugate to that which is composed of all the substitutions that are commutative to T' is that T and T' are conjugate.*

When an element of a substitution is replaced by itself it is said to form a cycle of a single element. *E. g.* the substitution *abc.de* is said to consist of two cycles if it is regarded as a substitution of degree five. If it is regarded as a substitution of degree six it is said to consist of three cycles, etc. According to this definition of cycle the product of any substitution s and a transposition† t involves either one more cycle or one less cycle than s. First suppose that the elements of t are found in two cycles of s and that each of these cycles begins with an element of t. It is clear that st contains a single cycle which includes all the elements of these two cycles and hence st has one less cycle than s. The other possibility is that the two elements of s are found in the same cycle of s and we may suppose that this cycle c begins with one of these elements. Let $c=a_1 \ldots a_\alpha \ldots a_\beta$ and $t=a_1 a_\alpha$ then will $ct=a_1 \ldots a_{\alpha-1} a_\alpha \ldots a_\beta$, the subscripts of the elements of c being arranged in ascending order and $\alpha \geqq 2$, $\beta \geqq \alpha$. Hence we observe that in this case st contains one more cycle than s.

Any cycle and hence any substitution can be represented as the product of a series of transpositions in an infinite number of ways. For instance, $c=a_1 a_2 a_3 \ldots a_l = a_1 a_2 \times a_1 a_3 \times \ldots \times a_1 a_l = a_1 a_3 \times a_3 a_2 \times a_1 a_4 \times \ldots a_1 a_l =$ etc.

By means of the results in the preceding paragraph we can readily prove that the numbers of these transpositions, for a given substitution s, are either all odd or all even. For, let $s=t_1 t_2 \ldots t_r$, t_1, t_2, ..., t_r being transpositions. Then will $st_r \ldots t_2 t_1 = 1$. If γ is the number of cycles of s and n its degree the first member of the last equation involves $n-\gamma$ more cycles than s. Hence $r-(n-\gamma)$ must be even; *i. e.* if one of the two numbers r, $n-\gamma$ is even the other is even also. Since the latter of these numbers is fixed when the substitution is given the theorem‡ is proved. Incidentally we observe that $n-\gamma$ is the smallest number of transposittons whose product is a substitution of degree n and γ cycles and that $r-(n-\gamma)$ can be made equal to any even integer. s is called positive or negative as r is even or odd.

Since the product of two positive substitutions is positive it follows that every group which involves negative substitutions must contain a subgroup composed of its positive substitutions. The order of this subgroup is half the order of the entire group since the product of a positive and a negative substitution is negative and the product of two negative substitutions is positive. The group which is composed of all the positive substitutions of degree n is called the alternating group of order $n!\div 2$. If a subgroup is transposed into itself by all the substitutions of the entire group it is called *selfconjugate*. A subgroup which includes half the substitutions of a group must be either selfconjugate or one of two conjugate subgroups. The latter is impossible since all the substitutions of one

*Two substitutions, or groups, are conjugate when one is the transform of the other.

†A transposition is a substitution of degree two; *e. g.*, *ab*, *cd*, ...

‡For a different proof of this fundamental theorem see Burnside, *Theory of Groups*, 1897, page 9; Cole's edition of *Netto's Theory of Substitution*, 1892, page 17; etc.

of these conjugates would have to transform the other into itself for a substitution must transform every subgroup in which it occurs into itself.

While every subgroup whose order is half the order of the group is self-conjugate a subgroup of any lower order need not be self-conjugate. If a group contains a subgroup of one-third its order that is not selfconjugate it must contain three subgroups of this order which are transformed according to the symmetric group of order six by all the substitutions of the group.

AN ELEMENTARY EXPOSITION OF GRASSMANN'S "AUSDEHNUNGSLEHRE," OR THEORY OF EXTENSION.

By JOS. V. COLLINS. Ph. D., Stevens Point, Wis.

[Concluded from November Number.]

APPLICATION TO MECHANICS.

168. A force is completely represented by a point vector. We will denote a force by $F\rho$, where F denotes the length and direction of a vector, indicating respectively the intensity and direction of the force, and ρ is the point of application. It is apparent here that two letters are needed to properly represent the complex concept of a force. (See 76).

158. In Chapter I, we saw that the sum of two vectors is the diagonal of a parallelogram whose adjacent sides are the two given vectors. Then the sum of two forces, or their resultant, is the diagonal of a parallelogram whose two adjacent sides represent the two given forces. Similarly, all the results obtained for vectors in that chapter hold equally well for forces. The condition for equilibrium of forces acting on a particle at the extremity of ρ is evidently $(\Sigma F)\rho=0$, or $\Sigma F=0$.

169. The formulas obtained in Chapter VI evidently hold true when the points are replaced by infinitesimal forces, as parallel forces acting on particles, and also when they are replaced by finite parallel forces. (See 80).

170. Let the resultant of the forces acting on a rigid body be denoted by R. Then if $\varepsilon=\rho-\rho_1$,

$$R=\Sigma F\rho=\Sigma F\rho_1+\Sigma F(\rho-\rho_1)=(\Sigma F)\rho_1+(\Sigma F\varepsilon).$$

This result, called a "Wrench," contains two parts, a vector and a plane segment part. The vector ΣF represents the translation force, and the plane segment $\Sigma F\varepsilon$ gives the plane and magnitude of rotation. When the above result is interpreted geometrically, *i. e.* when ΣF is thought of as a line, and $\Sigma F\varepsilon$ as a plane segment, R is called a "Screw."

171. If R reduce to a single force, then by 34, $R^2=0$. We have then

$$R^2=(\Sigma F\rho_1+\Sigma F\varepsilon)^2=2(\Sigma F.\Sigma F\varepsilon)\rho_1=0.$$

This shows that ΣF and $\Sigma F\varepsilon$ must be parallel (112), *i. e.* the resultant force is parallel to the plane of the resultant couple. Evidently $\Sigma F.\Sigma F\varepsilon=0$ is satisfied also either by $\Sigma F=0$ whence R=a couple, or by $\Sigma F\varepsilon=0$, which makes of R a single force. If all of the forces lie in a single plane, the resultant can be reduced to either a single force or to a couple.

172. *For equilibrium we must have* (170), $\Sigma F=0$ *and* $\Sigma F\varepsilon=0$.

173. *A total resultant effect may be reduced to a single force and a couple whose place is perpendicular to the force by properly choosing the point of application.*

PROOF.

$$R=\Sigma F\rho_1+\Sigma F(\rho-\rho_1)=\Sigma F\rho_2-\Sigma F(\rho_2-\rho_1)+\Sigma F(\rho-\rho_1).$$

Let $\Sigma F(\rho-\rho_1)=|\varepsilon$. Then

$$R=\Sigma F\rho_2-\Sigma F(\rho_2-\rho_1)+|\varepsilon.$$

Now the condition that $\Sigma F\rho_2$ shall be perpendicular to $|\varepsilon-\Sigma F(\rho_2-\rho_1)$ is

$$(|\varepsilon-\Sigma F(\rho_2-\rho_1)|\Sigma F\rho_2=|\varepsilon|\Sigma F\rho_2-\Sigma F(\rho_2-\rho_1).|\Sigma F\rho_2=0.$$

$\therefore |\Sigma F\rho_2.\Sigma(\rho_2-\rho_1)F=-|\Sigma F\rho_2\varepsilon$, (64, 38), whence

$$\frac{|\Sigma F\rho_2.\Sigma(\rho_2-\rho_1)F}{(\Sigma F\rho_2)^2}=-\frac{|\Sigma F\rho_2\varepsilon}{(\Sigma F\rho_2)^2}.$$

Comparing the left member of this equation with the formula of 132, we see that the right member is the value of the orthogonal projection of $\rho_2-\rho_1$ on a plane perpendicular to $\Sigma F\rho_2$.

Multiplying $\Sigma F\rho_2$ by the members of the last equation and interchanging the factors of $\Sigma(\rho_2-\rho_1)F$, we get

$$\Sigma F(\rho_2-\rho_1)=\frac{\Sigma F\rho_2|\Sigma F\rho_2\varepsilon}{(\Sigma F\rho_2)^2}=\frac{|\Sigma F\rho_2.\Sigma F\rho_2|\varepsilon}{(\Sigma F\rho_2)^2}-|\varepsilon \text{ (144, last equation).}$$

Substituting the last value of $\Sigma F(\rho_2-\rho_1)$ in the first equation of this article, we have

$$R=\Sigma F\rho_2+\frac{\Sigma F\rho_2\Sigma F(\rho_2-\rho_1)}{(\Sigma F\rho_2)^2}.|\Sigma F\rho_2,$$

which gives the required reduction.

CHAPTER XIV.

APPLICATION TO LOGIC.

174. The law of the inner product is $e_r e_s = 0$. For in 117 if F is different from E, $[E|F]$ contains equal factors and is therefore zero, (43). This law, which is the opposite of that of the outer product (34), is made the basis of the study of spaces. Now a space in the *Ausdehnungslehre* corresponds to a *concept* or notion in logic. Hence the former science can be applied in the latter.

175. DEFINITION.—The *Combined Space*, or the *sum* of two spaces, is the totality of quantities which belong to one or other of them (17).

176. DEFINITION.—The *Common Space*, or the *product* of two spaces, is the totality of quantities which are common to both, (see 104). The product of two spaces which have no quantity of the first order common is zero (174).

Thus, using $^\circ(L_1 L_2)$ and $^\circ(L_1 L_3)$ to denote the two spaces which contain these lines, we see that L_1 is the common space.

177. *The sum of the orders m and n of two spaces equals the sum of the orders p and q of their common and combined spaces.*

Evidently $m+n$ duplicates the number expressing the order of the common space, and $p+q$ does the same.

178. *All the laws of space analysis continue to hold true in logic when the word "space" is replaced by the logical term "concept."*

PROOF.—It is evident from the definitions 175, 176 that $^\circ(e+e) = {}^\circ e$, and $^\circ(e.e) = {}^\circ e$, and also that $^\circ e_r + {}^\circ e_s$ is greater or less than 0 and $^\circ(e_r).^\circ(e_s) = 0$, when r is greater or less than s. But these are likewise the basic formulas of logic. (See H. and R. Grassmanns' *Formenlehre*, B. II., *Die Begriffslehre, oder Logik*,

1872, page 43. See also Encyclopedia Britannica article 'Logic,' section 35, paragragh 6.)

179. Two spaces can contain equal simple quantities only when they overlap. Thus if $A = abcd$ and $A' = a'b'c'd'$, these spaces have $a'bcd'$ in common, and quantities in $a'bcd'$ are in both A and A'. But E and F have no simple quantities in common. Since they are of the same size and lie in the same plane E and F in *geometry* would be equal (94); but in the theory of space or logic they are altogether different, having not one point, element, or bit of surface in common. It is highly important to note this difference if the reader is to avoid misconception.

180. *The associative and commutative laws for addition and multiplication hold for space analysis.*

PROOF.—It is evident from the definition 175 that $a+b=b+a$. Similarly from the definition in 176 it follows that $ab=ba$, and $abc=a(bc)$.

181. *Every sum, n.°A, or product, $(°A)^n$, formed from the same space, °A, equals this space.* This follows from proof in 178.

182. *One can add to any space a product of two spaces, one of whose factors is the given space, without altering its value.* Thus

$$°A=°A+°A°B.$$

One can multiply any space by a sum of two spaces, one of whose parts is the given space, without altering its value. Thus,

$$°A=°A(°A+°B).$$

PROOF.—The product $°A°B$ is that part of $°A$ which is common to both factors, 176. But adding a part of $°A$ to $°A$ gives $°A$ by 181. In the other case we see that what is common to $°A$ and $°A+°B$ is $°A$.

183. *If* $°A+°B=°B$, $°A.°B=°A$.

PROOF.—$°A.°B=°A(°A+°B)$ (Hyp.)
$=°A$ (82)

If $°A.°B=°A$, $°A+°B=°B$.

PROOF.—$°A+°B=°A°B+B°$ (Hyp.)
$=°A°B+°B^2$ (181)
$=°B(°A+°B$ (Dist. Law)
$=°B$ (182)

184. *Unity added to any space gives unity, and any space multiplied by unity gives the same space; nought added to any space gives the same space, and any space multiplied by nought gives nought.*

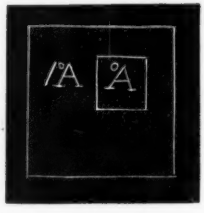
PROOF.—The last three of these results are self-evident. From the first we have

$$1=1(1+1\times°A) \qquad (182)$$
$$=1+°A \qquad \text{(Dist. law and 2. of this Article.)}$$

185. *If* $°A+°C=°B+°C$ *and also* $°A.°C=°B.°C$, $°A=°B$.

PROOF.—$°A=°A(°A+°C)$ (182)$=°A(°B+°C)$ (hyp.)$=°A.°B+°A°C$ (Dist. law)
$=°A°B+°B°C$(hyp.)$='B(°A+°C)=°B(°B+°C)$(hyp.)$=°B$ (182).

DEFINITION.—The Non-Space or Complementary Space of $°A$ is that space $|°A$ (read non-A) which contains all the units not found in $°A$, and none of the units which are found in $°A$.



187. *The sum of a space °A and $|°A$ is unity: the product of a space °A and $|A$ is zero.*

PROOF.—The truth of the second part of the theorem follows directly from 176. For the other we write, by 184,

$$°A+|°A=(°A+|°A)|.$$

But by 176 the product of two spaces cantains all the quantities which are common to both. Then all the quantities of the left member $°A+|°A$ are contained in $|$. But $°A$ and $|°A$ contain all the units there are. Then $|$ contains all possible units.

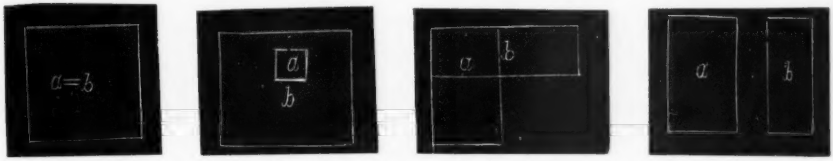
188. *All non-spaces of the same spaces are equal.*

PROOF.—Suppose $|^\circ A$ and $|^\circ A_1$ to be two non-spaces of $^\circ A$. Then $^\circ A+|^\circ A=1=^\circ A+|^\circ A_1$; whence by 185, remembering that $^\circ A.|^\circ A=0$ and $^\circ A.|^\circ A_1=0$, $|^\circ A=|^\circ A_1$.

189. *The non-space of the non-space of a given space is the given space.*

PROOF.—We have $|^\circ A+||^\circ A=1=|^\circ A+^\circ A$, and $||^\circ A.|^\circ A=0=^\circ A\,|^\circ A$(187). Whence, by 185, $||^\circ A=^\circ A$.

190. We give below four diagrams with descriptive names.



Identical Spaces. Incident Space. Cutting Spaces. Disjunctive Spaces.

191. *Each of two spaces is incident to their sum. The product of two spaces is incident to each of them. Unity is the highest space, to which all other spaces are incident Nought is the lowest space, being included in all other spaces.* Compare the use of unity here with its use in Chapter V.

The theorems already given will suffice to show that the language and subject matter of the *Ausdehnungslehre* can be utilized in the study of logic. The material for this chapter is taken from Robert Grassmann's *Ausdehnungslehre* (Slettin, 1891). Robert Grassmann aided his brother Hermann in the preparation of the *Ausdehnungslehre* of 1862 and the *Formenlehre* of 1872, and was always deeply interested in the subject.

The article in the Britannica already referred to and Professor Stokes' article in the January (1900) number of THE AMERICAN MATHEMATICAL MONTHLY show that there is considerable diversity of opinion regarding the philosophy underlying the application of mathematics to logic.

INTEGRATION OF ELLIPTIC INTEGRALS.

By G. B. M. ZEER. A. M., Ph. D., Professor of Chemistry and Physics. The Temple College. Philadelphia, Pa.

[Concluded from November Number.]

$$\int \frac{d\theta}{\cos^6\theta\sqrt{(1-e^2\sin^2\theta)}} = \frac{1}{1-e^2}\int \sqrt{(1-e^2\sin^2\theta)}\,\frac{d\theta}{\cos^6\theta}$$

$$-\frac{e^2}{1-e^2}\int \frac{d\theta}{\cos^4\theta\sqrt{(1-e^2\sin^2\theta)}} = \frac{(8-19e^2+15e^4)}{15(1-e^2)^2}F(e,\theta)$$

$$-\frac{(8-23e^2+23e^4)}{15(1-e^2)^3}E(e,\ \theta)+\frac{(8-23e^2+23e^4)}{15(1-e^2)^3}\tan\theta\sqrt{(1-e^2\sin^2\theta)}$$

$$+\frac{\sin\theta}{5(1-e^2)\cos^5\theta}\sqrt{(1-e^2\sin^2\theta)}+\frac{4(1-2e^2)}{15(1-e^2)^2}\cdot\frac{\sin\theta}{\cos^3\theta}\sqrt{(1-e^2\sin^2\theta)}=S_3 \quad \ldots\ldots(108).$$

From the foregoing we deduce the following relations :

$$V_2=e^2V_1+U_2, \quad V_3=e^2V_2+U_3.$$

$\therefore$ Generally, $V_n=e^2V_{n-1}+U_n$ ..... (109).

$$3V_2=2(1+e^2)V_1-e^2V_0-(\cos\theta/\sin^3\theta)\sqrt{(1-e^2\sin^2\theta)},$$

$$5V_3=4(1+e^2)V_2-3e^2V_1-(\cos\theta/\sin^5\theta)\sqrt{(1-e^2\sin^2\theta)}.$$

$\therefore$ Generally, $(2n-1)V_n=(2n-2)(1+e^2)V_{n-1}-(2n-3)e^2V_{n-2}$

$$-(\cos\theta/\sin^{2n-1}\theta)\sqrt{(1-e^2\sin^2\theta)} \quad \ldots\ldots(110).$$

$$(1-e^2)S_2=R_2-e^2S_1, \quad (1-e^2)S_3=R_3-e^2S_2.$$

$\therefore$ Generally, $(1-e^2)S_n=R_n-e^2S_{n-1}$ ..... (111).

$$3(1-e^2)S_2=2(1-2e^2)S_1-e^2S_0+(\sin\theta/\cos^3\theta)\sqrt{(1-e^2\sin^2\theta)},$$

$$5(1-e^2)S_3=4(1-2e^2)S_2-3e^2S_1+(\sin\theta/\cos^5\theta)\sqrt{(1-e^2\sin^2\theta)}.$$

$\therefore$ Generally, $(2n-1)(1-e^2)S_n=(2n-2)(1-2e^2)S_{n-1}$

$$-(2n-3)e^2S_{n-2}+(\sin\theta/\cos^{2n-1}\theta)\sqrt{(1-e^2\sin^2\theta)} \quad \ldots\ldots(112).$$

$$3U_2=(2-e^2)U_1-e^2(1-e^2)v_0-(\cos\theta/\sin^3\theta)\sqrt{(1-e^2\sin^2\theta)},$$

$$5U_3=(4-e^2)U_2-e^2(1-e^2)V_1-(\cos\theta/\sin^5)\sqrt{(1-e^2\sin^2\theta)}.$$

$\therefore$ Generally, $(2n-1)U_n=(2n-2-e^2)u_{n-1}-e^2(1-e^2)v_{n-2}$

$$-(\cos\theta/\sin^{2n-1}\theta)\sqrt{(1-e^2\sin^2)} \quad \ldots\ldots(113).$$

$$3(1-e^2)R_2=(2-e^2)R_1-e^2S_0+(\sin\theta/\cos^3\theta)\sqrt{(1-e^2\sin^2\theta)},$$

$$5(1-e^2)R_3=(4-3e^2)R_2-e^2S_1+(\sin\theta/\cos^5\theta)\sqrt{(1-e^2\sin^2\theta)}.$$

$\therefore$ Generally, $(2n-1)(1-e^2)R_n=[2n(1-e^2)-(2-3e^2)]R_{n-1}$

$$-e^2S_{n-2}+(\sin\theta/\cos^{2n-1}\theta)\sqrt{(1-e^2\sin^2\theta)} \quad \ldots\ldots(114).$$

The following integrals come under the above :

$$\int \sqrt{(a^4+2b^2x^2+x^4)}dx, \quad \int \sqrt{(a^4-2b^2x^2+x^4)}dx, \text{ where } a<b.$$

Let $b^2+(b^4-a^4)^{\frac{1}{2}}=m^2$, $b^2-(b^4-a^4)^{\frac{1}{2}}=n^2$.

$\therefore (a^4+2b^2x^2+x^4)=(x^2+m^2)(x^2+n^2)$ ..... (115).

$(a^4-2b^2x^2+x^4)=(x^2-m^2)(x^2-n^2)$ ..... (116).

In (115), let $x=n\tan\theta$. $\therefore dx=n\sec^2\theta d\theta$.

In (116), let $x=m\operatorname{cosec}\theta$. $\therefore dx=-m\operatorname{cosec}\theta\cot\theta d\theta$.

Let $e^2=(m^2-n^2)/m^2$, $k^2=n^2/m^2$.

$$\therefore \int \sqrt{(a^4+2b^2x^2+x^4)}dx=n^2\int \frac{\sqrt{[m^2-(m^2-n^2)\sin^2\theta]}d\theta}{\cos^4\theta}$$

$$=mn^2\int \frac{\sqrt{(1-e^2\sin^2\theta)}d\theta}{\cos^4\theta}=mn^2R_2, \text{ from (102).}$$

$$\int \sqrt{(a^4-2b^2x^2+x^4)}dx=-m^3\int \frac{\cos^2\theta\sqrt{(m^2-n^2\sin^2\theta)}d\theta}{\sin^4\theta}$$

$$=m^3\int \frac{\sqrt{(1-k^2\sin^2\theta)}d\theta}{\sin^2\theta}-m^3\int \frac{\sqrt{(1-k^2\sin^2\theta)}d\theta}{\sin^4\theta}=m^3(U_1-U_2), \text{ from (97), (101)}$$

Let $(3)=G_1$, $(5)=G_2$, $(10)=G_3$, etc. $(4)=H_1$, $(7)=H_2$, $(13)=H_3$, etc.

$(8)=I_1$, $(15)=I_2$, $(22)=I_3$, etc. $(9)=K_1$, $(16)=K_2$, $(25)=K_3$, etc.

and writing (e, θ) for $(e, \frac{1}{2}\pi)$ we get

$3e^2G_2=2(1+e^2)G_1-G_0$, $5e^2G_3=4(1+e^2)G_2-3G_1$.

$\therefore$ Generally, $(2n-1)e^2G_n=(2n-1)(1+e^2)G_{n-1}-(2n-3)G_{n-2}$ ..... (117).

$3e^2H_2=(1-e^2)H_0-2(1-2e^2)H_1$, $5e^2H_3=3(1-e^2)H_1-4(1-2e^2)H_2$.

$\therefore$ Generally, $(2n-1)e^2H_n=(2n-2)(1-e^2)H_{n-2}$

$$-(2n-1)(1-2e^2)H_{n-1} \text{ ..... (118).}$$

$I_1=G_1-e^2G_2$, $I_2=G_2-e^2G_3$, $I_3=G_3-e^2G_4$.

$\therefore$ Generally, $I_n = G_n - e^2 G_{n+1}$ .....(119).

$K_1 = (1-e^2)H_1 + e^2 H_2$, $K_2 = (1-e^2)H_2 + e^2 H_3$, $K_3 = (1-e^2)H_3 + e^2 H_4$.

$\therefore$ Generally, $K_n = (1-e^2)H_n + e^2 H_{n+1}$ .....(120).

$$\int \frac{d\theta}{\sin^2\theta \sqrt{(1-e^2\sin^2\theta)}} = -\frac{\cot\theta}{\sqrt{(1-e^2\sin^2\theta)}} + e^2 \int \frac{\cos^2\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}$$

$$\therefore \int \frac{\cos^2\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}} = \frac{\cot\theta}{e^2\sqrt{(1-e^2\sin^2\theta)}} + \frac{1}{e^2}\int \frac{d\theta}{\sin^2\theta \sqrt{(1-e^2\sin^2\theta)}}$$

$$= \frac{1}{e^2}F(e,\ \theta) - \frac{1}{e^2}E(e,\ \theta) + \frac{\sin\theta\cos\theta}{\sqrt{(1-e^2\sin^2\theta)}} = T_1 \quad ..(121).$$

$$\int \frac{d\theta}{\cos^2\theta \sqrt{(1-e^2\sin^2\theta)}} = \frac{\tan\theta}{\sqrt{(1-e^2\sin^2\theta)}} - e^2 \int \frac{\sin^2\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}.$$

$$\therefore \int \frac{\sin^2\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}} = \frac{\tan\theta}{e^2\sqrt{(1-e^2\sin^2\theta)}} - \frac{1}{e^2}\int \frac{d\theta}{\cos^2\theta \sqrt{(1-e^2\sin^2\theta)}}$$

$$= \frac{1}{e^2(1-e^2)}E(e,\ \theta) - \frac{1}{e^2}F(e,\ \theta) - \frac{\sin\theta\cos\theta}{(1-e^2)\sqrt{(1-e^2\sin^2\theta)}} = W_1 \quad .....(122).$$

$$\int \frac{d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}} = T_1 + W_1 = \frac{1}{1-e^2}E(e,\ \theta) - \frac{e^2\sin\theta\cos\theta}{(1-e^2)\sqrt{(1-e^2\sin^2\theta)}} = W_0 \,..(123_0).$$

$$\int \frac{\cos^2\theta d\theta}{\sin^2\theta \sqrt{(1-e^2\sin^2\theta)}} = -\frac{\cos^2\theta\cot\theta}{\sqrt{(1-e^2\sin^2\theta)}} - 2(1-e^2)\int \frac{\cos^2\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}$$

$$-e^2\int \frac{\cos^4\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}.$$

$$\therefore \int \frac{\cos^4\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}} = \frac{1}{e^2}\int \frac{d\theta}{\sqrt{(1-e^2\sin^2\theta)}} - \frac{1}{e^2}\int \frac{d\theta}{\sin^2\theta \sqrt{(1-e^2\sin^2\theta)}}$$

$$-\frac{2(1-e^2)}{e^2}\int \frac{\cos^2\theta d\theta}{(1-e^2\sin^2\theta)^2} - \frac{\cos^2\theta\cot\theta}{e^2\sqrt{(1-e^2\sin^2\theta)}} = \frac{(2-e^2)}{e^4}E(e,\ \theta)$$

$$-\frac{2(1-e^2)}{e^4}F(e,\ \theta) - \frac{1-e^2}{e^2}\cdot\frac{\sin\theta\cos\theta}{\sqrt{(1-e^2\sin^2\theta)}} = T_2 \,..(123).$$

$$\int \frac{\cos^4\theta d\theta}{\sin^2\theta \sqrt{(1-e^2\sin^2\theta)}} = -\frac{\cot\theta\cos^4\theta}{\sqrt{(1-e^2\sin^2\theta)}} - 4(1-e^2)\int \frac{\cos^4\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}$$

$$-3e^2\int \frac{\cos^6\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}.$$

$$\therefore \int \frac{\cos^6\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{7}{2}}} = \frac{2}{3e^2}\int \frac{d\theta}{\sqrt{(1-e^2\sin^2\theta)}} - \frac{1}{3e^2}\int \frac{d\theta}{\sin^2\theta\sqrt{(1-e^2\sin^2\theta)}}$$

$$-\frac{1}{3e^2}\int \frac{\sin^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}} - \frac{4(1-e^2)}{3e^2}\int \frac{\cos^4\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{7}{2}}} - \frac{\cot\theta\cos^4\theta}{3e^2\sqrt{(1-e^2\sin^2\theta)}}$$

$$= \frac{(8-17e^2+9e^4)}{3e^6}F(e,\ \theta) - \frac{(8-13e^2+3e^4)}{3e^6}E(e,\ \theta)$$

$$+\frac{(4-6e^2+3e^4)}{3e^4}\cdot\frac{\sin\theta\cos\theta}{\sqrt{(1-e^2\sin^2\theta)}} - \frac{\sin^3\theta\cos\theta}{3e^2\sqrt{(1-e^2\sin^2\theta)}} = T_3 \quad \ldots (124).$$

$$\int \frac{dx}{(a^4+2b^2x^2+x^4)^{\frac{3}{2}}} = \frac{1}{n^2}\int \frac{\cos^4\theta d\theta}{[m^2-(m^2-n^2)\sin^2\theta]^{\frac{3}{2}}} = \frac{1}{n^2\,m^3}\int \frac{\cos^4\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}$$

$$= \frac{T_2}{n^2\,m^3} \ldots . (125). \qquad \int \frac{dx}{(a^4-2b^2x^2+x^4)^{\frac{3}{2}}} = -\frac{1}{m^2}\int \frac{\sin^4\theta d\theta}{\cos^2\theta(m^2-n^2\sin^2\theta)^{\frac{3}{2}}}$$

$$= -\frac{1}{m^5}\int \frac{\sin^4\theta d\theta}{\cos^2\theta(1-k^2\sin^2\theta)^{\frac{3}{2}}} = \frac{2}{m^5}\int \frac{d\theta}{(1-k^2\sin^2\theta)^{\frac{3}{2}}}$$

$$-\frac{1}{m^5}\int \frac{d\theta}{\cos^2\theta(1-k^2\sin^2\theta)^{\frac{3}{2}}} - \frac{1}{m^5}\int \frac{\cos^2\theta d\theta}{(1-k^2\sin^2\theta)^{\frac{3}{2}}} = \frac{1}{m^5}(2W_0 - T_1)$$

$$-\frac{1}{m^5}\int \frac{d\theta}{\cos^2\theta(1-k^2\sin^2\theta)^{\frac{3}{2}}} = \frac{1}{m^5}(2W_0 - T_1) + \frac{k^2}{m^5(1-k^2)}\int \frac{d\theta}{(1-k^2\sin^2\theta)^{\frac{3}{2}}}$$

$$-\frac{1}{m^5(1-k^2)}\int \frac{d\theta}{\cos^2\theta\sqrt{(1-k^2\sin^2\theta)}} = \frac{1}{m^5}\left(\frac{2-k^2}{1-k^2}W_0 - \frac{1}{1-k^2}S_1 - T_1\right) \ldots (126).$$

In (126) and in the value of $\int \sqrt{(a^4-2b^2x^2+x^4)}dx$, k is written instead of e.

$$\int \frac{\sin^2\theta d\theta}{\cos^2\theta\sqrt{(1-e^2\sin^2\theta)}} = \frac{\tan\theta\sin^2\theta}{\sqrt{(1-e^2\sin^2\theta)}} - 2\int \frac{\sin^2\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}$$

$$+e^2\int \frac{\sin^4\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}. \qquad \therefore \int \frac{\sin^4\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}} = \frac{1}{e^2}\int \frac{d\theta}{\cos^2\theta\sqrt{(1-e^2\sin^2\theta)}}$$

$$-\frac{1}{e^2}\int \frac{d\theta}{\sqrt{(1-e^2\sin^2\theta)}} + \frac{2}{e^2}\int \frac{\sin^2\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}} - \frac{\tan\theta\sin^2\theta}{e^2\sqrt{(1-e^2\sin^2\theta)}}$$

$$= \frac{2-e^2}{e^4(1-e^2)}E(e,\ \theta) - \frac{2}{e^4}F(e,\ \theta) - \frac{\sin\theta\cos\theta}{e^2(1-e^2)\sqrt{(1-e^2\sin^2\theta)}} = W_2 \quad \ldots\ldots (127).$$

$$\int \frac{\sin^4\theta d\theta}{\cos^2\theta\sqrt{(1-e^2\sin^2\theta)}} = \frac{\tan\theta\sin^4\theta}{\sqrt{(1-e^2\sin^2\theta)}} - 4\int \frac{\sin^4\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}$$

$$+3e^2\int \frac{\sin^6\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}. \qquad \therefore \int \frac{\sin^6\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{7}{2}}} = \frac{1}{3e^2}\int \frac{d\theta}{\cos^2\theta\sqrt{(1-e^2\sin^2\theta)}}$$

$$-\frac{2}{3e^2}\int\frac{d\theta}{\sqrt{(1-e^2\sin^2\theta)}}+\frac{1}{3e^2}\int\frac{\cos^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}+\frac{4}{3e^2}\int\frac{\sin^4\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}$$

$$-\frac{\tan\theta\sin^4\theta}{3e^2\sqrt{(1-e^2\sin^2\theta)}}=\frac{8-3e^2-2e^4}{3e^6(1-e^2)}E(e,\ \theta)-\frac{(8+e^2)}{3e^6}F(e,\ \theta)$$

$$-\frac{(1-e^2)\sin\theta\cos\theta}{3e^4\sqrt{(1-e^2\sin^2\theta)}}-\frac{\sin\theta\cos^3\theta}{3e^4\sqrt{(1-e^2\sin^2\theta)}}=W_3\ \ldots\ (128).$$

Comparing (3) and (121) we get,

$$\int_0^{\frac{1}{2}\pi}\frac{\sin^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}=\int_0^{\frac{1}{2}\pi}\frac{\cos^2\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}.$$

Comparing (6) and (123) we get,

$$3\int_0^{\frac{1}{2}\pi}\frac{\sin^2\theta\cos^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}=\int_0^{\frac{1}{2}\pi}\frac{\cos^4\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}.$$

Comparing (12) and (124) we get,

$$5\int_0^{\frac{1}{2}\pi}\frac{\sin^2\theta\cos^4\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}=\int_0^{\frac{1}{2}\pi}\frac{\cos^6\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}.$$

$\therefore$ Generally, $(2n-1)\int_0^{\frac{1}{2}\pi}\frac{\sin^2\theta\cos^{2n-2}\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}=\int_0^{\frac{1}{2}\pi}\frac{\cos^{2n}\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}.$

Comparing (4) and (122) we get,

$$\int_0^{\frac{1}{2}\pi}\frac{\cos^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}=(1-e^2)\int_0^{\frac{1}{2}\pi}\frac{\sin^2\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}.$$

Comparing (6) and (127) we get,

$$3\int_0^{\frac{1}{2}\pi}\frac{\sin^2\theta\cos^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}=(1-e^2)\int_0^{\frac{1}{2}\pi}\frac{\sin^4\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}.$$

Comparing (11) and (128) we get,

$$5\int_0^{\frac{1}{2}\pi}\frac{\sin^4\theta\cos^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}=(1-e^2)\int_0^{\frac{1}{2}\pi}\frac{\sin^6\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}.$$

$\therefore$ Generally, $(2n-1)\int_0^{\frac{1}{2}\pi}\frac{\sin^{2n-2}\theta\cos^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}=(1-e^2)\int_0^{\frac{1}{2}\pi}\frac{\sin^{2n}\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}.$

$$\int\frac{d\theta}{\sin^2\theta(1-e^2\sin^2\theta)^{\frac{3}{2}}}=\int\frac{d\theta}{\sin^2\theta\sqrt{(1-e^2\sin^2\theta)}}+e^2\int\frac{d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}$$

$$=V_1+e^2W_0=L_1.$$

$$\int\frac{d\theta}{\cos^2\theta(1-e^2\sin^2\theta)^{\frac{3}{2}}}=\frac{1}{1-e^2}\int\frac{d\theta}{\cos^2\theta\sqrt{(1-e^2\sin^2\theta)}}-\frac{e^2}{1-e^2}\int\frac{d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}$$

$$=[1/(1-e^2)](S_1-e^2W_0)=O_2.$$

$$\int \frac{d\theta}{\sin^4\theta(1-e^2\sin^2\theta)^{\frac{3}{2}}} = \int \frac{d\theta}{\sin^4\theta\sqrt{(1-e^2\sin^2\theta)}} + e^2 \int \frac{d\theta}{\sin^2\theta(1-e^2\sin^2\theta)^{\frac{3}{2}}}$$

$$= V_2 + e^2 L_1 = L_2.$$

$$\int \frac{d\theta}{\cos^4\theta(1-e^2\sin^2\theta)^{\frac{3}{2}}} = \left(\int \frac{d\theta}{\cos^4\theta\sqrt{(1-e^2\sin^2\theta)}} - e^2 O_1\right)\left(\frac{1}{1-e^2}\right)$$

$$= [1/(1-e^2)](S_2 - e^2 O_1) = O_2.$$

$\therefore$ Generally, $\int \frac{d\theta}{\sin^{2n}\theta(1-e^2\sin^2\theta)^{\frac{3}{2}}} = V_n + e^2 L_{n-1} = L_n.$

$$\int \frac{d\theta}{\cos^{2n}\theta(1-e^2\sin^2\theta)^{\frac{3}{2}}} = \frac{1}{1-e^2}(S_n - e^2 O_{n-1}) = O_n.$$

The following general integrals may be of interest. The last term in the right hand member being the one sought.

$$\int \left(\frac{1}{\cos^2\theta} - \frac{1}{\sin^2\theta}\right) \frac{d\theta}{(1-e^2\sin^2\theta)^{\frac{1}{2}(2n-1)}} = \frac{\sec\theta\operatorname{cosec}\theta}{(1-e^2\sin^2\theta)^{\frac{1}{2}(2n-1)}}$$

$$+(2n-1)e^2 \int \frac{d\theta}{(1-e^2\sin^2\theta)^{\frac{1}{2}(2n+1)}}.$$

$$\int \frac{\cos^{2m-2}\theta d\theta}{\sin^2\theta(1-e^2\sin^2\theta)^{\frac{1}{2}(2n-1)}} = -\frac{\cot\theta\cos^{2m-2}\theta}{(1-e^2\sin^2\theta)^{\frac{1}{2}(2n-1)}}$$

$$-(2m-2)(1-e^2)\int \frac{\cos^{2m-2}\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{1}{2}(2n+1)}} - (2m-2n-1)e^2 \int \frac{\cos^{2m}\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{1}{2}(2n+1)}}.$$

$$\int \frac{\sin^{2m-2}\theta d\theta}{\cos^2\theta(1-e^2\sin^2\theta)^{\frac{1}{2}(2n-1)}} = \frac{\tan\theta\sin^{2m-2}\theta}{(1-e^2\sin^2\theta)^{\frac{1}{2}(2n-1)}}$$

$$-(2m-2)\int \frac{\sin^{2m-2}\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{1}{2}(2n+1)}} + (2m-2n-1)e^2 \int \frac{\sin^{2m}\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{1}{2}(2n+1)}}.$$

$$\int \frac{d\theta}{\sin^{2m}\theta(1-e^2\sin^2\theta)^{\frac{1}{2}(2n-1)}} + e^2 \int \frac{d\theta}{\sin^{2m-2}\theta(1-e^2\sin^2\theta)^{\frac{1}{2}(2n+1)}}$$

$$= \int \frac{d\theta}{\sin^{2m}\theta(1-e^2\sin^2\theta)^{\frac{1}{2}(2n+1)}}.$$

$$\frac{1}{1-e^2}\int \frac{d\theta}{\cos^{2m}\theta(1-e^2\sin^2\theta)^{\frac{1}{2}(2n-1)}} - \frac{e^2}{1-e^2}\int \frac{d\theta}{\cos^{2m-2}\theta(1-e^2\sin^2\theta)^{\frac{1}{2}(2n+1)}}$$

$$= \int \frac{d\theta}{\cos^{2m}\theta(1-e^2\sin^2\theta)^{\frac{1}{2}(2n+1)}}.$$

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

Aprops of the ending of a century, the following is said to have been written on New Year's Day, 1801, by Theodore Dwight (1764—1846), the brother of Timothy Dwight, the distinguished Congregational divine.

"Precisely 12 o'clock last night
The 18th century took its flight.
Full many a calculating head
Has racked its brains, its ink has shed
To prove by metaphysics fine
A hundred means but ninety-nine;
While at their wisdom others wondered,
But took one more to make a hundred.
Strange at the 18th century's close
While light in beams effulgent glows,
When bright illumination's ray
Has chased the darkness far away,
Heads filled with mathematics lore
Dispute if two and two make four.
Go on, ye scientific sages,
Collect your light a few more ages,
Perhaps as swells the vast amount
A century hence you'll learn to count."

December 10, 1900. M. A. GRUBER.

133. Proposed by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tennessee.

In Wentworth's Arithmetic he gives a formula $\frac{2}{4}\frac{1}{0}(d^2-2d)$ for calculating the number of board feet in a log 10 feet long, when d is the diameter in inches. How is this rule derived?

No solution of this problem has been received.

135. Proposed by NELSON L. RORAY, Bridgeton, N. J.

If 6 is one-half of 10, what part of 20 is 12? Also what part of 30 is 10?

Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.; MARTIN SPINKS, Wilmington, O.; P. S. BERG, B. S., Larimore, N. D.; G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.; DANIEL B. NORTHROP, Mandana, N. Y.

The general statement of this problem is, If a is nb, c is dx. What is the value of x? $a:nb=c:dx$.

$\therefore x=\frac{nbc}{ad}$.

Substituting the numerical values, $x=\frac{\frac{1}{2}\times 10\times 12}{6\times 20}=\frac{1}{2}$.

$\therefore$ 12 is one-half of 20, if, etc.

Also $x=\frac{\frac{1}{2}\times 10\times 10}{6\times 30}=\frac{5}{18}$.

$\therefore$ 10 is $\frac{5}{18}$ of 30, if, etc.

PROOF. $\frac{1}{2}$ of $10=5$; $\frac{1}{2}$ of $20=10$; and $\frac{5}{18}$ of $30=8\frac{1}{3}$.

$6:5=12:10$; also, $6:5=10:8\frac{1}{3}$.

GEOMETRY.

SOME PROPOSITIONS ON THE REGULAR DODECAHEDRON.

By ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, University, Miss.

(1). The edges are parallel, in pairs.

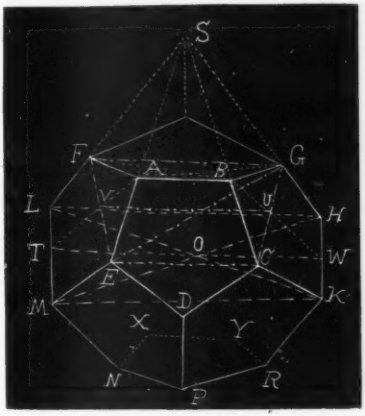
To prove HK parallel to LM.

Since the faces are equal regular pentagons FG and EC are equal.

They are also parallel, each being parallel to AB.

Hence GC and FE are equal and parallel.

Therefore HK, which is parallel to GC, and LM, which is parallel to FE, are parallel.



(2). The planes of the faces are parallel, in pairs.

To prove the plane of face AG parallel to the plane of face NR.

As in (1) GC was proved parallel to FE.

So may FB be proved parallel to MD, and, therefore, parallel to NP.

Similarly, AG and PR are parallel.

Therefore, the planes of faces NR and AG are parallel.

(3). Edges in different faces extending from vertices of the same face are inclined to each other at angles of 60°.

To prove HG and LF inclined to each other at an angle of 60°.

EA and CB produced intersect at S.

From the similar triangles SAB and SEC, $SA:AB=SE:EC$.

Representing an edge of the dodecahedron by a, and a diagonal of a face by d, this proportion gives $SA=\frac{a^2}{d-a}$.

But $d=\frac{a}{2}[\sqrt{5}+1]$. Therefore, $SA=\frac{a}{2}[\sqrt{5}+1]=d$.

That is, CB produced meets EA produced at a point distant d from A.

Similarly LF produced meets EA produced at a point distant d from A; hence, at S.

In this way it can be shown that HG produced passes through S.

Hence LF and HG are in the same plane; and triangle SFG is equilateral, each side being equal to d.

Therefore LF and HG are inclined to each other at an angle of 60°.

(4). If the ten diagonals of two non-adjacent, non-parallel, faces being grouped in pairs, both faces being represented in every pair, (a) the diagonals of one pair are parallel, (b) those of each of two pairs meet, when produced,

at angles of 60°, and (*c*) those of each of two other pairs are inclined to each other at angles of 36°.

Consider the faces BK and AM.

(*a*). CG is parallel to EF (proved in the demonstration of (1)).

(*b*). KB and MA lie in the same plane.

For AB is parallel to FG; and FG is parallel to LH, since FG divides the sides of the triangle SLH proportionally; and LH is parallel to MK, since LM and HK and equal and parallel.

Hence AB and MK are parallel and determine a plane.

KB and MA are parallel to HG and LF, respectively.

Therefore, when, produced, they meet at an angle of 60°.

Similarly with HB and LA.

(*c*). KG and MF are in the same plane, since FG and MK are parallel.

And they are parallel to CB and EA, respectively.

The angle between them, therefore, equals angle ASB, or 36°.

So with HC and LE.

(5). To express the diagonals in terms of a and d.

One of the shortest is EG, one of the longest is LK, and a medium diagonal is LH.

Denote these by s, l, and m, respectively.

LH is a side of the equilateral triangle SLH.

Hence, $m=SF+FL=\frac{a}{2}[\sqrt{5}+1]+a$

$$=\frac{a}{2}[\sqrt{5}+3]=\frac{1}{a}\left[\frac{a}{2}[\sqrt{5}+1]\right]^2=\frac{d^2}{a}.$$

To determine s and l it is necessary to show that $LMKH$ is a rectangle, and $FECG$ a square.

The plane bisecting LM at right angles contains all points equidistant from L and M, and is perpendicular to all lines parallel to LM.

Therefore this plane will pass through the middle point of FE, through points A and B, and, hence, the line AB, through the middle points of GC and HK, intersecting the planes LK and FC in the lines TW and UV, respectively.

Since AB is parallel to FG and LH, it is parallel to planes FC and LK, and, hence, to UV and TW.

FG, UV, LH, and TW, then, are parallel, and since FE and LM are respectively perpendicular to UV and TW, they are perpendicular to FG and LH respectively.

It follows that FC is a square and LK a rectangle.

From right triangle ECG, $s=\frac{a\sqrt{2}}{2}[\sqrt{5}+1]=\sqrt{2}\,d.$

From right triangle LMR, $l=\frac{a\sqrt{3}}{2}[\sqrt{5}+1]=\sqrt{3}\,d.$

This result may be written $\sqrt{\frac{a}{2}[3+\sqrt{5}]3a}=\sqrt{3am}.$

It may be remarked that d is a mean proportional between a and m, l is a mean proportional between $3a$ and m, and l is the diagonal of a cube whose face-diagonal is s and edge d.

(6). To determine a point equidistant from all the vertices.

LK bisects MH at O.

In the rectangle of which GH and MN are opposite sides, GN bisects MH, therefore passes through O and is itself bisected at O.

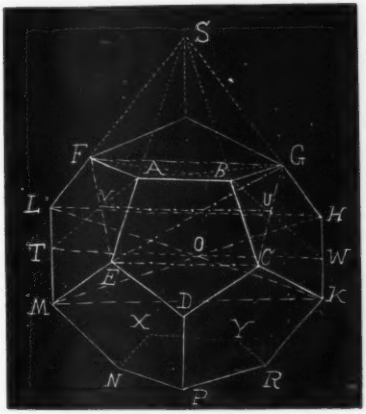
So with all the longest diagonals.

Hence O is equidistant from all the vertices.

COROLLARY 1. O is equidistant from all the edges.

COROLLARY 2. O is equidistant from all the faces.

(7). To compute the volume.



This could be done by conceiving the dodecahedron composed of twelve equal pyramids with O as their common vertex and the faces for bases. But another method will be adopted.

The dodecahedron is composed of a cube one of whose faces is $EFGC$ and six equal truncated triangular prisms, one of which has AB, FG, and EC for its lateral edges. A right section of this truncated prism is a triangle with base equal to EF, and altitude, the perpendicular from A to UV. Since $ABXY$ is a rectangle equal to $LMKH$, this perpendicular is one-half the difference between $AX(=LH)$ and an edge of the cube, and, therefore, equals $\frac{1}{2}a$.

Hence the area of the right section is $\frac{a^2}{8}[\sqrt{5}+1]$, and the volume of the truncated prism is $\frac{a^2}{8}[\sqrt{5}+1]\frac{a+a[\sqrt{5}+1]}{3}$.

The sum of the volumes of six such solids is $\frac{a^3}{4}[7+3\sqrt{5}]$.

Volume of cube$=\frac{a^3}{4}[8+4\sqrt{5}]$.

Volume of dodecahedron$=\frac{a^3}{4}[15+7\sqrt{5}]$.

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134. Proposed by J. C. GREGG, A. M., Superintendent of Schools, Brazil, Ind.

If $ABCD$ is a quadrilateral circumscribing a circle, show that the line joining the middle points of the diagonals AB, CD passes through the center of the circle.

Solution by HARRY S. VANDIVER, Bala, Penna.

We will use quadrilinear coördinates denoting the equation of the four

sides $\alpha=0$, $\beta=0$, $\gamma=0$, $\delta=0$, and the *lengths* of the corresponding sides by a, b, c, and d. Let radius of circle be r. Then the equation of the line joining the middle points of the diagonals is $a\alpha-b\beta+c\gamma-d\delta=0$....(1) (Cf. Salmon's *Conics*, page 54, Ex. 5).

Putting $\alpha=\beta=\gamma=\delta=r$ we obtain $r(a-b+c-d)=0$, which is satisfied since $a+c=b+d$.

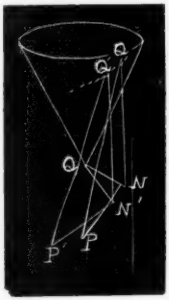
137. Proposed by J. W. YOUNG, Oliver Graduate Scholar in Mathematics, Cornell University, Ithaca, N. Y.

A right cone has its vertex in a horizontal plane, its axis being perpendicular to the plane. A string has one extremity attached to a point on the cone. The other extremity, P, of the string is kept in the plane, and the string is then wound around the cone, without being allowed to slip. Show that the spiral generated by P cuts all straight lines through the vertex at the same angle.

Solution by the PROPOSER.

Let P, P' be two points on the spiral; Q, Q' the corresponding points in the path of the string around the cone; N, N' the points where the perpendiculars from Q, Q' to the plane through the vertex O of the cone, cut the plane.

The right-angled triangles QNO, $Q'N'O$ have the angles QON and $Q'ON'$ equal; hence they are similar.



$$\therefore \frac{QN}{ON}=\frac{Q'N'}{ON'}\ldots.(1).$$

Again, since the string must not slip, it makes a constant angle with the plane.

$\therefore$ $\triangle QNP$ is similar to $\triangle Q'N'P'$.

$$\therefore \frac{PN}{QN}=\frac{P'N'}{Q'N'}\ldots.(2).$$

From (1) and (2),

$$\therefore \frac{PN}{ON}=\frac{P'N'}{ON'}\ldots.(3).$$

But the triangles ONP and $ON'P'$ are right-angled at N and N' (PN, $P'N'$ being the projections to tangents to the circular cone). From (3),

$\therefore$ $\triangle ONP$ is similar to $\triangle ON'P'$.

$\therefore$ $\measuredangle OPN=\measuredangle OP'N'$.

Observing that PN and $P'N'$ are normals to the spiral, the last equation states that the normals make a constant angle with rays through O. Q. E. D.

AVERAGE AND PROBABILITY.

90. Proposed by WALTER H. DRANE, Graduate Student, Harvard University, Cambridge, Mass.

During a heavy rain storm a circular pond is formed in a circular field. If a man undertakes to cross the field in the dark, what is the chance that he will walk into the pond? [From *Byerly's Integral Calculus.*]

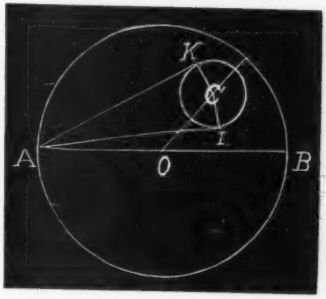
I. Solution by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

Let O be the center of the circular field, and R its radius; C the center of

the circular pond and z its radius; $x=OC$, the distance from the center of the field to the center of the pond; $\angle ROC=\theta$, $\angle IAC=\phi$.

Then $AC=(R^2+x^2+2Rx\cos\theta)^{\frac{1}{2}}$ and $\phi=\sin^{-1}\left(\frac{z}{\sqrt{(R^2+x^2+2Rx\cos\theta)}}\right)$.

Then the probability that the man will walk into the pond for any particular value of ϕ is



$$\frac{2\phi.AK}{\pi AK}=\frac{2\phi}{\pi}.$$

The limits of z are 0 and $R-x$; of x, 0 and R; and of θ, 0 and π, and doubled.

Hence, the probability that the man walks into the pond is

$$p=\frac{2}{\pi}\frac{\int_0^{\pi}\int_0^{R}\int_0^{R-x}\sin^{-1}\left(\frac{z}{\sqrt{(R^2+x^2+2Rx\cos\theta)}}\right)d\theta.2\pi x dx.dz}{\int_0^{\pi}\int_0^{R}\int_0^{R-x}d\theta.2\pi x dx.dz}$$

$$=\frac{12}{\pi^2R^3}\int_0^{\pi}\int_0^{R}\int_0^{R-x}\sin^{-1}\left(\frac{z}{\sqrt{(R^2+x^2+2Rx\cos\theta)}}\right)d\theta.xdx.dz$$

$$=\frac{12}{\pi^2R^3}\int_0^{\pi}\int_0^{R}\left[z\sin^{-1}\left(\frac{z}{(R^2+x^2+2Rx\cos\theta)^{\frac{1}{2}}}\right)\right.$$

$$\left.+\sqrt{1-\frac{z^2}{(R^2+x^2+2Rx\cos\theta)}}\right]_0^{R-x}d\theta.xdx$$

$$=\frac{12}{\pi^2R^3}\int_0^{\pi}\int_0^{R}\left[(R-x)\sin^{-1}\left(\frac{R-x}{\sqrt{(R^2+x^2+2Rx\cos\theta)}}\right)+2\sqrt{(Rx)}\cos\tfrac{1}{2}\theta\right.$$

$$\left.-\sqrt{(R^2+x^2+2Rx\cos\theta}\right]d\theta.xdx$$

$$=\frac{12}{\pi^2R^3}\int_0^{\pi}\int_0^{R}(R-x)\tan^{-1}\left(\frac{R-x}{2\sqrt{(Rx)}\cos\frac{1}{2}\theta}\right)+\frac{12}{\pi^2R^3}\int_0^{\pi}\int_0^{R}[2\sqrt{(Rx)}\cos\tfrac{1}{2}\theta$$

$$-\sqrt{(R^2+x^2+2Rx\cos\theta)}]d\theta.xdx$$

$$=\frac{12}{\pi^2R^3}\int_0^{\pi}\left(\left[\tfrac{1}{6}x^2\tan^{-1}\left(\frac{R-x}{2\sqrt{(Rx)}\cos\frac{1}{2}\theta}\right)\right]_0^{R}\right.$$

$$\left.+\tfrac{1}{6}\int_0^{R}\frac{x^{\frac{3}{2}}R^{\frac{1}{2}}(R+x)(3R-2x)\cos\frac{1}{2}\theta}{R^2+x^2+2Rx\cos\theta}dx\right)d\theta$$

$$+\frac{12}{\pi^2R^3}\int_0^\pi\Big[\tfrac{4}{5}R^{\frac{1}{2}}x^{\frac{5}{2}}\cos\tfrac{1}{2}\theta-\tfrac{1}{3}(R^2+x^2+2Rx\cos\theta)^{\frac{3}{2}}+\tfrac{1}{2}R\cos\theta(x+2R\cos\theta)$$

$$\sqrt{(R^2+x^2+2Rx\cos\theta)}+\tfrac{1}{2}R^3\sin^2\theta\cos\theta\log[\sqrt{(R^2+x^2+2Rx\cos\theta)}+x+R\cos\theta]\Big]_0^R d\theta$$

$$=\frac{2}{\pi^2R^3}\int_0^R\int_0^\pi\frac{x^{\frac{3}{2}}R^{\frac{1}{2}}(R+x)(3R-2x)\cos\tfrac{1}{2}\theta}{R^2+x^2+2Rx\cos\theta}dxd\theta+\frac{12}{\pi^2R^3}\int_0^\pi\frac{R^3}{30}\Big[24\cos\tfrac{1}{2}\theta$$

$$-140\cos^3\tfrac{1}{2}\theta+120\cos^5\tfrac{1}{2}\theta+10-15\cos^2\theta+15\sin^2\theta\cos\theta\log\Big(\frac{1+\cos\frac{1}{2}\theta}{\cos\frac{1}{2}\theta}\Big)\Big]d\theta$$

$$=\frac{2}{\pi^2R^3}\int_0^R\Big[\frac{1}{\sqrt{-1}}x(3R-2x)\tan^{-1}\Big(\frac{2\sqrt{(-Rx)}}{R+x}\sin\tfrac{1}{2}\theta\Big)\Big]_0^\pi dx+\frac{2}{5\pi^2}\int_0^\pi\Big[24\cos\tfrac{1}{2}\theta$$

$$-140\cos^3\tfrac{1}{2}\theta+120\cos^5\tfrac{1}{2}\theta+10-15\cos^2\theta+15\sin^2\theta\cos\theta\log\Big(\frac{1+\cos\frac{1}{2}\theta}{\cos\frac{1}{2}\theta}\Big)\Big]d\theta$$

$$=\frac{2}{\pi^2R^3}\int_0^R\frac{1}{\sqrt{-1}}\Big[x(3R-2x)\tan^{-1}\Big(\frac{2\sqrt{(-Rx)}}{R+x}\Big)\Big]dx-\frac{32}{\pi^2}$$

$$=\frac{2}{\pi^2R^3}\Big(\Big[\frac{x^2}{6\sqrt{-1}}(9R-4x)\tan^{-1}\Big(\frac{2\sqrt{(-Rx)}}{R+x}\Big)\Big]_0^R$$

$$-\frac{\sqrt{(-R)}}{6\sqrt{(-1)}}\int_0^R\frac{x^{\frac{3}{2}}(9R-4x)}{R-x}dx\Big)-\frac{32}{\pi^2}=\frac{5}{3\pi^2\sqrt{-1}}\tan^{-1}(\sqrt{-1})$$

$$-\frac{1}{3\pi^2R^{\frac{3}{2}}}\int_0^R(4x^{\frac{3}{2}}-5Rx^{\frac{1}{2}}-5R^2x^{-\frac{1}{2}}+\frac{5R^2}{\sqrt{x}(R-x)})dx-\frac{32}{\pi^2}$$

$$=\frac{5}{3\pi^2\sqrt{-1}}\tan^{-1}(\sqrt{-1})-\frac{1}{3\pi^2R^{\frac{3}{2}}}\Big[\tfrac{8}{5}x^{\frac{5}{2}}-\tfrac{10}{3}Rx^{\frac{3}{2}}-10R^2x^{\frac{1}{2}}$$

$$+\frac{5R^{\frac{3}{2}}}{\sqrt{-1}}\tan^{-1}\sqrt{\Big(-\frac{x}{R}\Big)}\Big]_0^R-\frac{32}{\pi^2}=\frac{176}{45\pi^2}-\frac{32}{15\pi^2}=\frac{16}{9\pi^2}.$$

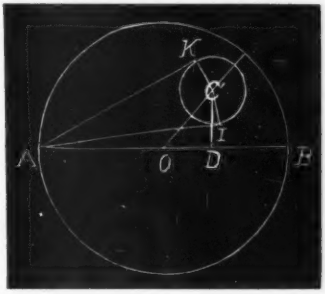
J. M. Colaw, M. E. Graber, and Walter H. Drane get the result $p=l/L$, where l is the circumference of the pond, and L the circumference of the field. But the circumference of the pond varies from 0 to $2\pi R$ where R is the radius of the field. Assuming as they do, that the chance of the man walking into the pond is the totality of all lines crossing both the field and pond divided by the totality of all lines crossing the field, the result l/L is not satisfactory, since l is variable, as already stated.

The problem, of course, is indefinite, since the law of variation of the several events are not definitely stated, and thus as many different results may be obtained as there are possible interpretations of the meaning of the problem.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let O be the center of the field, radius r; C the center of the pond, radius c. The chance p, of walking into the pond is $\angle KAI/\pi=\varphi/\pi$.

Let $(x-a)^2+(y-b)^2=c^2$ be the equation to the pond; $y=m(x+r)$, the equation to a line meeting the pond.



When $y=m(x+r)$ is tangent to the pond we have

$$\tan BAE=m_2=-\frac{b(a+r)+c\sqrt{b^2-c^2+(a+r)^2}}{c^2-(a+r)^2}$$

$$\tan BAF=m_1=-\frac{b(a+r)-c\sqrt{b^2-c^2+(a+r)^2}}{c^2-(a+r)^2}$$

$$\therefore \tan EAF=\frac{m_1-m_2}{1+m_1m_2}=\frac{2c\sqrt{[b^2-c^2+(a+r)^2]}}{[b^2-c^2+(a+r)^2]-c^2}=\tan\varphi.$$

$$\therefore \varphi=\tan^{-1}\left(\frac{2c\sqrt{[b^2-c^2+(a+r)^2]}}{[b^2-c^2+(a+r)^2]-c^2}\right)=2\tan^{-1}\left(\frac{c}{\sqrt{[b^2-c^2+(a+r)^2]}}\right).$$

If $a=\rho\cos\theta$, $b=\rho\sin\theta$, $\varphi=2\tan^{-1}\left(\frac{c}{\sqrt{(\rho^2+r^2+2\rho r\cos\theta)}}\right)$.

$$\therefore p=\frac{\frac{1}{\pi}\int_0^{\pi}\int_0^{r}\int_0^{r-\rho}\varphi\rho\,d\theta\,d\rho\,dc}{\int_0^{\pi}\int_0^{r}\int_0^{r-\rho}\rho\,d\theta\,d\rho\,dc}=\frac{6}{\pi^2 r^3}\int_0^{\pi}\int_0^{r}\int_0^{r-\rho}\varphi\rho\,d\theta\,d\rho\,dc$$

$$=\frac{12}{\pi^2 r^3}\int_0^{\pi}\int_0^{r}\left[(r-\rho)\tan^{-1}\left(\frac{r-\rho}{2\sqrt{(r\rho)}\cos\frac{1}{2}\theta}\right)+2\sqrt{(r\rho)}\cos\tfrac{1}{2}\theta-\sqrt{(\rho^2+r^2+2r\rho\cos\theta)}\right]\rho\,d\theta\,d\rho.$$

$$\frac{12}{\pi^2 r^3}\int_0^{\pi}\int_0^{r}[2\sqrt{(r\rho)}\cos\tfrac{1}{2}\theta-\sqrt{(\rho^2+r^2+2r\rho\cos\theta)}]\rho\,d\theta\,d\rho$$

$$=\frac{2}{5\pi^2}\int_0^{\pi}\left[24\cos\tfrac{1}{2}\theta-140\cos^3\tfrac{1}{2}\theta+120\cos^5\tfrac{1}{2}\theta+10-15\cos^2\theta+15\sin^2\theta\cos\theta\log\left(\frac{1+\cos\frac{1}{2}\theta}{\cos\frac{1}{2}\theta}\right)\right]d\theta=-\frac{32}{15\pi^2}.$$

$$\frac{12}{\pi^2 r^3}\int_0^{\pi}\int_0^{r}(r-\rho)\tan^{-1}\left(\frac{r-\rho}{2\sqrt{(r\rho)}\cos\frac{1}{2}\theta}\right)\rho\,d\theta\,d\rho$$

$$= \frac{2}{\pi^2 r^3}\int_0^{\pi}\int_0^{r} \frac{\rho\sqrt{(r\rho)}(r+\rho)(3r-2\rho)\cos\frac{1}{2}\theta\, d\rho\, d\theta}{\rho^2+r^2+2r\rho\cos\theta}.$$

$$= \frac{2}{\pi^2 r^3\sqrt{-1}}\int_0^{r}\rho(3r-2\rho)\tan^{-1}\left(\frac{2\sqrt{(-r\rho)}}{r+\rho}\right)d\rho.$$

$$\frac{2}{\pi^2 r^3\sqrt{-1}}\int_0^{r}\rho(3r-2\rho)\tan^{-1}\left(\frac{2\sqrt{(-r\rho)}}{r+\rho}\right)d\rho = \frac{5}{3\pi^2\sqrt{-1}}\tan^{-1}\sqrt{-1}$$

$$-\frac{1}{3\pi^2 r^3}\int_0^{r}\frac{\sqrt{(r\rho)}(9r\rho-4\rho^2)d\rho}{r-\rho} = \frac{5}{3\pi^2\sqrt{-1}}\tan^{-1}\sqrt{-1}$$

$$-\frac{1}{3\pi^2 r^3}\left[\tfrac{8}{5}\rho^2\sqrt{(r\rho)}-\tfrac{16}{3}r\rho\sqrt{(r\rho)}-10r^2\sqrt{(r\rho)}+\frac{5r^3}{\sqrt{-1}}\tan^{-1}\sqrt{-\frac{\rho}{r}}\right]_0^{r}$$

$$= \frac{5}{3\pi^2\sqrt{-1}}\tan^{-1}\sqrt{(-1)}+\frac{176}{45\pi^2}-\frac{5}{3\pi^2\sqrt{-1}}\tan^{-1}\sqrt{(-1)}=\frac{176}{45\pi^2}.$$

$$\therefore\ p=\frac{176}{45\pi^2}-\frac{32}{15\pi^2}=\frac{16}{9\pi^2}.$$

EDITORIALS.

The Summer Meeting, 1901, of the American Mathematical Society will be held at Cornell University.

Several Complete sets of the MONTHLY are wanted. Any reader possessing a complete set in good condition and desiring to sell it, should write to the editor stating the price desired.

The MONTHLY is mailed on the 28th of each month, and should reach the greater part of its readers within five days after it is mailed. Subscribers should notify us promptly of any change of address, or of any failure to receive their copies.

Dr. G. A. Miller has just been awarded the prize of $260. from the Cracow Academy of Austria for his work in the Theory of Groups. We rejoice with Dr. Miller in this well merited recognition of his work. Dr. Miller is a young mathematician, yet his contributions to *The American Journal of Mathematics*, *The Annals of Mathematics*, *The Messenger of Mathematics*, *The Proceedings of the London Mathematical Society*, *The Transactions of the American Mathematical Society*, the MONTHLY, and many other mathematical journals in both hemispheres, are very numerous.

Preliminary Announcement of the Courses of Study for the Summer Session, 1901, of Cornell University, has just been issued. During the session Dr. G. A. Miller will offer courses in Advanced Integral Calculus, History of Mathematics, and Introduction to the Theory of Groups and the Theory of Numbers.

BOOKS AND PERIODICALS.

Elements of Algebra. By James M. Taylor, A. M., LL. D., Professor of Mathematics, Colgate University. 8vo. Half Leather. 461 pages. Boston: Allyn & Bacon.

This book on Algebra is in keeping with the very excellent work on the Elements of the Calculus by the same author. Professor Taylor aims in this book at simplicity in method of presentation and at a natural and logical sequence in the series of steps which lead the student from his arithmetical experiences through his algebra, and in this, I think, he has succeeded admirably. Special attention is given to factoring, as it is the fundamental principal in the solution of quadratic and higher equations. The student who receives his preparatory mathematical training in a book of this character will have nothing to unlearn as he advances in his mathematical course. B. F. F.

The Elements of Astronomy. By Sir Robert Ball, LL. D., F. R. S., Lowndean Professor of Astronomy and Geometry in the University of Cambridge, and formerly Astronomer Royal of Ireland. 8vo. Cloth, viii+183 pages. Price, 80 cents. London and New York: The Macmillan Co.

The author, briefly but interestingly, traces the early beginnings of Astronomy in the introduction. He then devotes Chapter I. to Diurnal Motion; Chapter II., The Sun; Chapter III., The Apparent Motion of the Sun; Chapter IV., The Moon; Chapter V., Gravitation; Chapter VI., Mercury and Venus; Chapter VII., Mars: Chapter VIII., The Asteroids; Chapter IX., Jupiter; Chapter X., Saturn; Chapter XI., Uranus and Neptune; Chapter XII., Comets; Chapter XIII., Shooting Stars; Chapter XIV., Stars and Nebulæ; Chapter XV., The Causes Affecting the Apparent Places of the Stars. There are eleven excellent plates in addition to numerous diagrams. For a brief, scientific exposition of Astronomy, this little book is superior. B. F. F.

The Principles of Mechanics. An Elementary Exposition for the Students of Physics. By Frederick Slate, Professor of Physics in the University of California. Part I. 12mo. Cloth, x+299 pages. Price, $1.90. New York: The Macmillan Co.

"The material contained in these chapters has taken on its present form gradually, by a process of recasting, and sifting. The ideas guiding that process have been three: First, to select the subject matter with close reference to the needs of college students; second, to bring the instruction into adjustment with the actual stage of their training; and, third, to aim continually at treating Mechanics as a system of organized thought, having a clearly recognizable culture value."—*Preface.* It is great satisfaction to note the emphasis that modern writers on various branches of mathematics are laying on the culture value of the study of mathematics. The work before us is one worthy the consideration of teachers desiring a first-class text on Mechanics. B F. F.

Calculus with Applications. An Introduction to the Mathematical Treatment of Science. By Ellen Hayes, Professor of Applied Mathematics in Wellesley College. 8vo. Cloth, vii+162 pages. Price, $1.20. Boston and Chicago: Allyn & Bacon.

This little treatise presents in an admirable way the practical application of the calculus to some of the problems of the natural sciences. To those who wish, for the purpose of culture, to know in a simple and direct way, what the Calculus is and what it is for and also for those who are engaged in work in Chemistry, Astronomy, Economics, etc., this book is well adapted. B. F. F.

Differential and Integral Calculus. With Applications. For Colleges, Universities, and Technical Schools. By E. W. Nichols, Professor of Mathematics in the Virginia Military Institute, and author of Nichols's Analytical Geometry. 8vo. Half-Leather. xi+394 pages. Boston and Chicago: Allyn & Bacon.

This book is based upon the methods of "limits" and of "rates." Both methods are valuable; the former, however, has the advantage of greater rigor. Among the very commendable features of the book are: First, a large amount of explanations; second, clear and simple demonstrations of principles; third, geometrical, mechanical, and electrical applications; fourth, historical notes at the heads of chapters, giving a brief account of the discovery and development of the subject of which it treats; and fifth, foot notes calling attention to topics of special historic interest. B. F. F.

Esercizi ed Applicazioni di Trigonometria Piana, con 400 Escercizi e Problemi Propositi. Dal Prof. Christoforo Alasia. Milano, Italia: Ulrico Hoepli.

This book is a 12mo. cloth binding, containing 291 pages. It treats of the elements of Plane Trigonometry. Also some space is devoted to the discussion of "limits," "maxima" and "minima," and simultaneous trigonometric equations. There are thirty diagrams and 400 exercises. B. F. F.

Plane Trigonometry. By W. P. Durfee, Professor of Mathematics in Hobart College. 105 pages. Price, 80 cents. Boston: Ginn & Co. 1900.

This work covers the essential parts of Plane Trigonometry. The preliminary chapter on Computation is valuable. Stress is laid on the value of graphic methods in obtaining rough estimates; the limits of accuracy are pointed out, and the necessity of frequent checking is insisted on. The demonstrations are brief and they are illustrated by numerous exercises. The book admirably meets the needs of schools desiring a short course which is at the same time accurate. J. M. C.

Complete Trigonometry. By Webster Wells, S. B., Professor of Mathematics in the Massachusetts Institute of Technology. 171 pages. Price, 90 cents. Boston: D. C. Heath & Co. 1900.

The present volume is a revision of the author's Essentials of Trigonometry. Many improvements have been made, notably in the proofs of several of the functions, in the general demonstration of the formulae, in the solution of the right triangles by natural functions, and in the addition of a large number of examples. This is a brief but most satisfactory text. J. M. C.

Famous Geometrical Theorems and Problems. By William W. Rupert, C. E., Superintendent of Schools, Pottstown, Pa. 27 pages. Price, 10 cents. Boston: D. C. Heath & Co. 1900

This is No. 1 in the series of Heath's Mathematical Monographs, issued under the general editorship of Professor Webster Wells. It contains the history of several famous

theorems with demonstrations under each case. Twenty-six independent demonstrations are given of the Pythagorean theorem. The book will give much pleasure to teachers interested in the subject. J. M. C.

Mensuration. By S. A. Furst. 71 pages. Price, 50 cents. Harrisburg, Pa.: R. L. Myers & Co. 1899.

This little volume is intended to be supplemental to text-books both on arithmetic and geometry. The special feature of the work is the application of the prismoidal formula to finding the volume of solids. The work has many interesting problems and will doubtless meet with approval. J. M. C.

The Hall Arithmetics, (1) Elementary; (2) Complete. By Frank H. Hall. 248 and 448 pages. Prices, 35 cents and 60 cents. New York and Chicago: The Werner School Book Co. 1899.

The prominent features of the Werner Arithmetics have been preserved in this series, but better adaption for use in ungraded schools has been secured. The books are progressive and practical in character, and abundant in the supply of concrete problems. In the treatment prominence is given the "magnitude idea," and the elements of algebra and geometry have been judiciously introduced. J. M. C.

New Practical and *New Higher Arithmetics.* By A. W. Rich, Ph., B., Associate Professor of Mathematics in the Iowa State Normal School. 222 and 320 pages. Prices, 50 cents and 75 cents, respectively. Chicago: A. Flanagan Co. 1900.

The *Practical* is intended for use in grammar grades. Special features are a set of tables and drills for mental work; a presentation of definitions and principles in compact form; model problem solutions; and a great variety of test work. In the *Higher* book the mathematical signs are systematically presented, the model solution feature is carried forward, and larger place is given to definitions and principles. Numerous exercises and problems are given in all the different subjects. J. M. C.

Hornbrook's Grammar School Arithmetic. By A. R. Hornbrook, A. M. 416 pages. Price, 65 cents. American Book Co. 1900.

This book is designed for the last four years of grammar school work, and aims to develop in the pupil a ready skill in dealing with numbers. In many respects this book differs from the ordinary texts. Much use is made of constructive work with simple geometric forms. The book will repay examination. J. M. C.

We are indebted to Prof. Alexander Macfarlane, D. Sc., LL. D., Lecturer on Mathematical Physics at Lehigh University, for a copy of *Space-Analysis*, a brief of twelve lectures on the George Leib Harrison Foundation, delivered in College Hall, University of Pennsylvania, February 5 to March 2, 1900.

ERRATA.

[Due to errors in the copy.]

Vol. VII, No. 11 (November, 1900), page 240, in the diagram,

for $\frac{\partial W}{\partial y}dy$, read $\frac{\partial W}{\partial y}dy=\delta$; for $i\frac{\partial V}{\partial y}dy$, read $-i\frac{\partial V}{\partial y}dy$; for idy, read $-idy$.

Page 241, line 7 from top,

$$\text{for } \frac{\partial V}{\partial y}dy = -\frac{\partial U}{\partial x}dx, \text{ read } \frac{\partial V}{\partial y} = -\frac{\partial U}{\partial x}.$$

INDEX.

ALGEBRA (See Solutions and Problems).
ARITHMETIC (See Solutions and Problems).
BIOGRAPHIES—
Sylvester Robins, by the Editor 179
Robert Tucker, by George Bruce Halsted 237–239
BOOKS, Reviewed 51-58, 88-90, 120-122, 148-150, 180, 203-206, 236, 275-276, 300-301
Elements of Precise Surveying and Geodesy, by Mansfield Merriman, 57; *Statistical Methods with Special Reference to Biological Variation*, by C. B. Davenport, 58; *Transactions of the American Mathematical Society*, edited by E. H. Moore, E. W. Brown, and T. S. Fiske, 58; *Synthetic Arithmetic*, by M. S. Cook, 88; *Supplement to Advanced Arithmetic*, by A. W. Plummer, 89; *Primary Exercises in Arithmetic*, by H. J. Silver, 89; *Solution Book*, by J. T. Fairchild, 89; *Advanced Arithmetic*, by Wm. W. Speer, 89; *Advanced Arithmetic*, by E. McN. Carr, 89; *Primary Arithmetic*, by E. McN. Carr, 89; *Essentials of Arithmetic*, by G. A. Southwork, 89; *The Public School Mental Arithmetic*, by J. A. McLellan, 90; *American Elementary Arithmetic*, by M. A. Bailey, 90; *Plane Trigonometry and Tables*, by D. A. Murray, 90; *The Teaching of Elementary Mathematics*, by D. E. Smith, 120; *Plane and Solid Geometry*, Revised Edition, by G. A. Wentworth, 121; *Algebra for Schools*, by G. W. Evans, 121; *Infallible Logic*, by Th. D. Hawley, 121; *Rational Elementary Arithmetic*, by H. H. Belfield, 148: *First and Second Books in Arithmetic*, by F. M. Wiemer and M. A. Bailey, 149; *Manual of Experimental Physics*, by F. R. Nichols, C. H. Smith, and C. M. Turton, 149; *Nine Ninety-Nine Graded Problems in Arithmetic*, by F. V. Lester, 149; *Lessons in Arithmetic*, by C. L. Howard, 149; *Graded Lessons in Arithmetic*, by W. F. Nichols, 149; *Gay's Problems in Arithmetic*, 149; *First Steps in Arithmetic*, by E. M. Pierce, 149; *The Werner Arithmetic*, by F. H. Hall, 149; *Rand-McNally Primary Arithmetic*, by E. C. Hewett, 150; *Graded Work in Arithmetic*, by S. W. Caird, 150; *Lippincott's Arithmetic*, by J. M. Rawlins, 150; *The American Monthly Review of Reviews*, by A. Shaw, 150; *Plane and Solid Geometry*, by A. A. Dodd and B. F. Chase, 180; *A Brief History of Mathematics*, by W. W. Beman and D. E. Smith, 180; *Elements of Algebra*, by W. W. Beman and D. E. Smith. 180; *Theory and Practice of Interpolation*, by H. L. Rice, 203; *School Arithmetic*, by J. M. Colaw and J. K. Ellwood, 203; *An Elementary Physics for Secondary Schools*, by C. B. Thwing, 203; *Higher Algebra*, by J. T. Downey, 204; *Holden's Elementary Astronomy*, 204; *History of English Literature*, by F. V. N. Painter, 204; *Plane Trigonometry with Tables*, by E. A. Lyman and E. C. Goddard, 204; *Essentials of Plane and Solid Geometry*, by W. Wells, 204; *Mental Arithmetic*, by E. Weidenhamer, 204; *The New Complete Arithmetic*, by D. M. Sensenig and R. F. Anderson, 205; *The Elements of Arithmetic*, by E. M. Pierce, 205; *The Wooster Arithmetic*, 205; *The Elements of the Differential and Integral Calculus*, by J. W. A. Young and E. E. Linebarger, 205; *The Gospel According to Darwin*, by W. Hutchi. son, 205; *School Arithmetic*, by J. M. Colaw and J. K. Ellwood, 236; *Popular Astronomy*, by W. W. Payne and H. C. Wilson, 236; *Logarithmic and Trigonometric Tables*, by D. A. Murray, 275; *Notes on the History of American Text-Books on Arithmetic*, by J. M. Greenwood and A. Martin, 276; *Elements of Algebra*, by James M. Taylor, 301; *The Elements of Astronomy*, by Sir Robert Ball, 301; *The Principles of Mechanics*, by Frederick Slate, 301; *Calculus with Applications*, by Ellen Hayes, 302; *Differential and Integral Calculus*, by E. W. Nichols, 302; *Esercizi ed Applicazioni di Trigonometria Piana*, by Christo-

foro Alasia, 302; *Plane Trigonometry*, by W. P. Durfee, 302; *Complete Trigonometry*, by Webster Wells, 302; *Famous Geometrical Theorems and Problems*, by Wm. W. Rupert, 302; *Mensuration*, by S. A. Furst, 303; *The Hall Arithmetics*. by F. H. Hall, 303; *New Practical* and *New Higher Arithmetics*, by A. W. Rich, 303; *Hornbrook's Grammar School Arithmetic*, by A. R. Hornbrook, 303.

DIOPHANTINE ANALYSIS (See Solutions and Problems).
EDITORIALS 30, 57, 148, 202, 235-236, 275, 300-301
ERRATA 206-309
GEOMETRY (See Solutions and Problems.)
MATHEMATICAL PAPERS—
Baker, A. L., Diagrammatic Proof of the Condition of Functionality in Complex Functions 240-242
Mathematical Induction 35-37
Collins, J, V., An Elementary Exposition of Grassmann's "Ausdehnungslehre," or Theory of Extension....31-35, 163-166, 181-187, 207-314, 253-258, 281-285
Emch, Arnold, On the Projectivity of Stresses in a Plane 134-135
Halsted, Dr. G. B., Gauss and the Non-Euclidean Geometry 247-252
Non-Euclidean Geometry 123-133
The Geometric, Old and New 154-155
The International Congress of Mathematicians 188-189
Heal, W. E., Expression of Riemann's P. Function as aDefinite Integral.... 155-166
Metzler, Prof. W. H., On a Determinant, Each of Whose Elements is the Product of k Factors 151-153
Miller, Dr. G. A., Examples of a few Elementary Groups 9-13
A Popular Account of Some New Fields of Thought in Mathematics 91-99
Some Methods of Constructing Substitution Groups 277-281
Miller, Hugh A., Twenty-Fifth Anniversary of Pedagogic Activity of Vasilief, 215-216
Roe, Dr. E. D., On the Transcendental Form of the Resultant 59-66
Stevens, James S., Forecasting the Census Returns 160-163
Stokes, G. J., Theory of Mathematical Inference 1-8
Whitaker. H. C., An Elementary Derivation of the Series for $\sin x$ and $\cos x$, 99-100
Zerr, G. B. M., Integration of Elliptical Functions 67-69, 100-102, 218-222, 242-246, 285-291
MAGIC SQUARES 228-229
MISCELLANEOUS (See Solutions and Problems).
NOTES—
Note on Problems 118 and 119, Arithmetic, by the Editor, 17; Note on Problem 84, Calculus, by Dr. E. Woelfing, Stuttgart, Germany, 19; Note on Problem 98, Algebra, by the Editor, 40; Note on Problem 118, by W. H. Carter, 44; Note on Prize Problem 86, by the Editor, 48; Note on Problem 118, by Geo. R. Dean, 78; Note on Problem 95, by Florian Cajori, 111-112; Note on the Solar Eclipse of May 28th, by the Editor, 147; Note on Prize Problem 104, Algebra, by the Editor, 172; Note on Problem 129, Arithmetic, by the Editor, 190; Note on the General Equation of the Second Degree, by Geo. R, Dean, 217-218; Note on Problem 133, Geometry, by the Editor, 227-228; Note on the Pythagorean Proposition, by Prof. D. A. Lehman, 228; Note on Problem 110, Algebra, by B. F. Yanney, 263.
PERIODICALS 58, 90, 150, 205-206, 236, 276

QUERY. When $a=b$, find value of $\dfrac{ab[-b\pm\sqrt{2a^2-b^2}]}{a^2-b^2}$.

SOLUTIONS AND PROBLEMS.

ARITHMETIC.

Balls in a box, Nos. 120 and 130 ..... 14, 70, 103, 189–190
Bought 150 head of stock for $300, paying for each kind a certain price. Find number of each kind, No. 126 135–136
Building and Loan Association Problem, No. 127 167–168
Circles, three equal, No. 125 103–104
Clock Problem, No. 124 70–72
Close of the Nineteenth Century 292
Figure 7 between hour and minute hands, No. 128 167–168
Frustrum, conical, to be divided into n parts, No. 131 223–224
If m cents be interest for M cents for p days, find yearly rate, No. 123 38
If 6 is half of 10, what part of 20 is 12? 292
Marble, Iron, Copper, Silver, Gold balls. Problem (in rhyme), No. 129 168–169
Road, 60 feet wide, crosses a square acre of land, West line passes through southwest corner of the land, east line through northeast corner. What fraction of the land in the road? No. 132 258–260
Rule for finding the number of board feet in round timber. How obtained, 292
Suppose 10% traction stock is 20% better in the market than 5% mining stock; if my income is $500. from each, how much have I paid, the investment bringing 6⅔%? No. 122 38
Three men with their sons, No. 121 14–15

ALGEBRA.

Farmer had 2080 lbs. of grain at the depot, etc., vid. problem No. 97 17–18
Field to be reaped by A and B for 90s, etc., vid problem No. 98 39–40
Given equation $x^m+p_1x^{m-1}+\ldots p_m$ freed from multiple roots. Prove discriminant positive or negative according as number of pairs of complex roots is even or odd. No. 103 137–138
Given $xyz=18$, $x^2+y^2+z^2=33$, $(x^2-yz)^3+(y^2-xz)^3+(z^2-xy)^3-3(x^2-yz)(y^2-zx)(z^2-xy)=6561$. No. 107 224–225
Prize Problem, $x^2+y=2$, $x+y^2=6$. No. 104 169–172
Prove $\Sigma(n)+\frac{n}{2!}\Sigma(n^2)+\frac{n(n-1)}{3!}\Sigma(n^3)+\ldots=(n+1)^2-n$. No. 101 104–107
Put down any number of £, s, and d under £11, etc., vid. problem No. 110 261–263
Seven persons to be seated at a round table. No. 99 72–75
Solve $a^{\log(a^x)}=m^{\log m}$. No. 105 172
Solve $\frac{x^2+x}{y^2+y}=a$, $\frac{x^2+y}{y^2+x}=b$. No. 106 190–192
Solve $\frac{1}{\sqrt{(x+1)}}+\frac{1}{\sqrt{(x-1)}}=\frac{1}{\sqrt{(x^2-1)}}$. No. 109 260–261

GEOMETRY.

Bisectors, two, of a triangle being equal, prove triangle isosceles. No. 133 226–228
Conic passing through all foci of a conic is a rectangular hyperbola. No. 124 42–43
Construct triangle, base, median to the base, and difference of base angles being given. No. 136 264–266
Cone, right, has its vertex in horizontal plane, its axis perpendicular to the plane, and string wound spirally on cone, etc. No. 90 296–300
Draw circle, cutting two circles orthogonally. No. 132 193–194
Equation to three points of conjugate diameters of an ellipsoid. No. 127 107–108

Given $F=\triangle^{n-1}\div(n-1)!\ \triangle_1.\triangle_2.\ldots\triangle_n$, etc., vid problem. No. 128 108–109
Hyperbola, touching four sides of an inscribed quadrilateral. If one focus lies on the circle, the other will lie on the circle, too. No. 135 264
Locus of a point on ellipsoid, the tangent plane at that point at distance f from center. No. 125 75–76
Medians, point of intersection of being G, prove $AB^2+BC^2+CA^2=3(GA^2+GB^2+GC^2)$. No. 123 41–42
Perpendiculars, dropped from vertices of a regular polygon upon any diameter. Prove sum on one side equals sum on other side. No. 122 ..... 40–41, 75
Points x, y, z divide strokes $c—b$, $a—c$, $b—a$, in same ratio r, and triangles x, y, z and a, b, c are similar, either $r=1$, or both triangles are equilateral. No. 130 172–173
Propositions on the regular dodecahedron 293–295
Prove $\lambda+\mu\omega+\nu\omega^2$ represents points of a quilt formed by regular hexagons. No. 131 173–174
Quadrilateral $ABCD$, circumscribing a circle. Line joining the middle points of diagonals AB and CD passes through center. No. 134 ..... 229–230, 295
Show that on no point of an ellipse will circle of curvature pass through center, eccentricity $<\frac{1}{2}\sqrt{2}$. No. 129 138–139
Through any fixed point O draw two straight lines at right angles. Let one line cut a given circle at Q, the other at R. Find locus of foot of perpendicular from O upon the chord QR. No. 126 77–78

CALCULUS.

Area of loop $r^2\cos\theta=a^2\sin^3\theta$. No. 91 19–20
A 24-inch joint of 6-inch stove pipe is compressed at one end to make it fit over an elliptical opening in a stove, etc., vid. problem. No. 101 231–232
Axes of three equal right cylinders intersect at right angles. Find common volume. No. 99 194
Cylinders, two intersecting. Find common volume. No. 92 20–22
Differential equation $(1-x^2)\frac{\partial^2 y}{\partial^2 x}-x\frac{\partial y}{\partial x}+y=2x$. No. 93 44–47
Epicycloid and hypocycloid. No. 98 139–140
If $f'(x)=\int f(x)\partial x$, find $f(x)$. No. 96 80
Minimum isosceles triangle about ellipse. No. 94 47
Park, elliptical, surrounded by wall. Horse fastened by a rope to wall; find area grazed over. No. 103 267–268
Prolate spheroid pierced by auger. Find volume removed. No. 97 109–110
Right cone has its vertex at focus of a paraboloid. Find common volume. No. 102 266–267
Ship, sailing northeast all the time. Find distance at given latitude and given longitude. No. 95 79–80
Volume, bounded by surface, generated by circumference of a circle, whose diameter is hypotenuse of a right triangle. No. 100 194–195

MECHANICS.

Assuming that velocity is proportionate to distance described from state of rest. Can a body start in motion, and if so, what is the initial acceleration? No. 95 82
Bow of a boat, inclined at a certain angle. When in motion the water was found to rise b inches on the bow. Find velocity of boat. No. 91 24
Fall from Heaven of the impious spirits. No. 87, ... 80–82, 110

Moments, principal, of a parallelogram. No. 94 and 97 ..... 48–49, 195
Particle slides down a parabola. Find initial velocity so that it may leave curve at extremity of latus rectum. No. 92 24–25
Planets, being supposed to have been originally one mass, are separated by an explosion, and force varying directly as distance; to prove the simultaneous colision of the planets. No. 90 22–24
Principal moment of inertia of triangle at centroid. No. 99 270–271
Rope passes over pulley, having at one end a given weight, and at other end hangs a monkey of a given weight. Find condition of impossibility of the mankey's climbing to the pulley. No. 93 48
Spool placed on a rough table. Vid. problem. No. 98 196
Two particles, subject to mutual attraction, and fixed center, move in a plane containing that center. Find the motion under the law of inverse square. No. 96 174–175

DIOPHANTINE ANALYSIS.

Arrange the consecutive integers 1 to n^2 as a magic square, when n is odd. Apply when $n=9$. No. 75 25–26
Four positive numbers to be found such that if each be diminished by twice the cube of their sum the four remainders will be rational cubes. No. 76 49–50
In series $1^3+3^3+5^3+\ldots$ find n so that the nth term and sum of n terms shall both be squares. No. 82 166
Integral right triangle in which the bisector of one of the acute angles is always integral. No. 79 83–85
Three square numbers in harmonic progression. No. 78 82–83
Three square numbers whose reciprocals form an arithmetical progression. No. 80 112–113

AVERAGE AND PROBABILITY.

Average area of a quadrilateral formed by joining the extremities of two chords perpendicular to each other and passing through a point at distance a from center of a circle radius R, No. 82 52–53
Average area of all ellipses of given major axis, No. 83 53–54
Box contains 100 balls marked from 1 to 100. 13 balls are drawn. Find chance that balls marked from 1 to 10 are among the 13. No. 80 27
From a point in the circumference of a circle two chords are drawn; find (1) average radius; and (2) averape area of circle touching the two chords and given circle. No. 84 113–114
Inch auger-hole bored through sphere. Find average volume of hole. No. 89, 197–198
Mean distance of all points on side of equilateral triangle from opposite vertex, and average length of line drawn across triangle. No. 81 50–52
Mean distance of random point in a sphere from a point, first, within; second, without, sphere. No. 87 141
Pond, circular, formed during a dark night in a circular field. Find chance of man walking into pond 296
Tetrahedron, average volume of, formed by four random points in a sphere. No. 88 176–177
Two equal spheres touch each other externally. If a point be taken within each sphere, show that, first, the distance between the points is less than diameter of either sphere is 13/55, and, second, the average distance between them, 11/52. No. 79 26–27
Two points are taken at random in a circle and a chord drawn through them; a point is taken at random in each segment. Find average area of

quadrilateral formed by joining the four points. No. 85 114–115
Two points are taken at random in a circular annulus of two concentric circles. Find chance that the straight line joining the two points will not cut the inner variable circle. No. 86 140–141

MISCELLANEOUS.

Cask, in shape of frustrum of spheroid, lies in a horizontal position. Find quantity of water in it. No. 81 178–179
Base-ball pitcher's curves to be explained. No. 77 142
Center of a regular polygon of n sides moves along a diameter of circle, its magnitude varying so that one of diagonals always coincides with a chord. Find volume and surface generated. No. 78 142–143
Ellipse seen as circle. No. 74 ..... 54-55, 115–116
Ice cream freezer, In an, cream of a homogeneous, etc., vid. problem. No. 73 36
Latitude λ°, In, a tree$=a$ leans in a direction $=\beta$, with an angle of elevation$=\gamma$. The sun's declination$=\delta$. Find direction of shadow when sun is on meridian. No. 79 144–145
Notes, Two, one for a dollars at $m\%$, etc., vid. problem. No. 86 272
Prove $e^{\pi} + e^{-\pi}=2[1+2^2][1+(\frac{2}{5})^2]$. . . ad infinitum. No, 84 199–200
Rhombuses, Rational, wherein area is by 1 less than the square of its sides. No. 80 145
Rational right triangle has at least one side divisible by 5. No. 85 ..... 232-233, 271
Segment's area of a circle being given, find chord. No. 83 199
Show $\log[x-a-b\sqrt{-1}]=\frac{1}{2}\log[(x-a)^2+b^2]-\sqrt{-1}.\tan^{-1}[b/(x-a)]$ No. 76 118–119
Spheres, Two equal, in contact. Find magnitude of sphere in contact with them. No. 82 232
Water tank at the Nacogdoches River, etc., vid. problem. No. 75 116–118

INDEX.

ALGEBRA (See Solutions and Problems).
ARITHMETIC (See Solutions and Problems).
BIOGRAPHIES—
Craig, Professor Thomas, by Professor F. P. Matz 183–187
Gauss, Karl Frederick, by B. F. Finkel 25–31
Hermite, Charles, by Dr. George Bruce Halsted 107–210
Newton, Sir Isaac, by Virginia Craig 157–161
Schmidt, Franz, by Dr. George Bruce Halsted 131–133
BOOKS, ... 24, 55-56, 79-80, 105-106, 129-130, 181-183, 211-212, 240
ALGEBRA. *Higher Algebra*, by Fisher & Schwatt, 187; *College Algebra*, by J. H. Boyd, 182.
ARITHMETIC. *The Common Sense of Commercial Arithmetic*, by George Hall, 79; Teachers' *Manual of School Arithmetic*, by J. M. Colaw and J. K. Elwood, 105.
ASTRONOMY. *Elements of Astronomy*, by Simon Newcomb, 129.
ENGINEERING. *Field Manual of Engineering*, by P. H. Philbrick, 24.
GEOMETRY. *The Elements of Analytical Geometry*, by A. L. Candy, 24; *Non-Euclidean Geometry*, by Henry P. Manning, 79; *Plane and Solid Analytical Geometry*, by Chas. H. Ashton, 106; *Geometric Exercises in Paper Folding*, by Sundara Row, 212; *Elementary Plane and Solid Geometry*, by Thomas F. Holgate, 240; *How to Attack an Exercise in Geometry*, by E. S. Loomis, 182.
GROUPS. *Linear Groups*, by Dr. L. E. Dickson, 212.
MECHANICS. *Theoretical Mechanics*. by W. Woolsey Johnson, 182.
PHYSICS. *Elements of Physics*, by Henry A. Rowland and Joseph S. Ames, 77; *Free Expansion of Gases*, Memoirs by Gay-Lussac, and others, 80; *A Manual of Laboratory Physics*, by H. M. Tory and F. H. Pritcher, 106; *One Hundred Problems in Physics*, by E. P. Thompson, 182; *The Elements of Physics*, by Henry Crew, 240.
TRIGONOMETRY. *Plane and Spherical Trigonometry*, by Elmer A. Lyman and Edwin C. Goodard, 240.
MISCELLANEOUS BOOKS. *The Teaching of Mathematics in the Higher Schools of Prussia*, by J. W. A. Young, 55; *Theory of Thought and Knowledge*, by Borden P. Bowne, 80; *On Continuity of Irrational Numbers and the Nature and Meaning of Numbers*, by Richard Dedekind, 105; *The Riddle of the Universe*, by Ernst Haeckel, 122; *The Measurement of General Exchange-Value*, by C. M. Walsh, 130; *A Short History of the Greeks*, by Evelyn S. Shuckburgh, 130; *Eléments de Mathématiques Supérieur*, by B. Vogt, 212.
DIOPHANTINE ANALYSIS (See Solutions and Problems).
EDITORIALS 23-24, 55, 78, 105, 129, 156, 181, 239, 268
ERRATA 24, 36, 106
GEOMETRY (See Solutions and Problems).
MATHEMATICL PAPERS—
Baker, A. Latham; Kinetic Derivation of Tangent Equations 111–115
Reduced Numbers 163–166
Lucus A Non Lûcendo 140–141
Barbarin, P.; On the Utility of Studying Non-Euclidean Geometry 161–163
Conoscente, Euplio; A Problem and its Solution 133–136
Dean, George R.; Note on Poles and Polars 81–83